International Journal of Theoretical Physics, Vol. 44, No. 12, December 2005 (© 2005)
DOI: 10.1007/s10773-005-8012-4

Decision Effects, the Hahn–Jordan Decomposition for States, and Their Connection to Ludwig's Axiomatic Approach to Quantum Mechanics

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We begin with a quantum logic carrying a large collection of states. We then form a dual pair of Banach spaces—base normed and order unit normed—containing the states and the logic, respectively. A Galois connection on the face lattices of the states and the dual positive order unit interval is introduced. The elements of the logic are connected to a dense subset of the extreme points of this order interval in the order unit space using a generalized form of the Hahn–Jordan decomposition theorem. Decision effects are defined and identified with the elements of the original logic. Finally, an important axiom of Ludwig is introduced which ties together all the lattices of Galois closed faces of states, Galois closed order intervals of the positive order unit interval, decision effects, and the original quantum logic. The emphasis here is on the consequences of functional analytic assumptions. The paper concludes with a simple example where Ludwig's axiom does not hold and we see parts of the theory dissolve.

KEY WORDS: quantum logic; effects; Ludwig's axiomatic development; base-normed spaces; order unit normed spaces; orthomodular posets.

PACS: 02.30.Sa; 03.65.Ta.

1. INTRODUCTION

In 1985, Professor Günther Ludwig published "An Axiomatic Basis for Quantum Mechanics, Volume 1, Derivation of Hilbert Space Structure" (Ludwig, 1985). We quote from the preface: "We seek to deduce the fundamental concepts of quantum mechanics solely from a description of macroscopic devices. The microscopic systems such as electrons, atoms, etc. must be detected on the basis of the macroscopic behavior of the devices." This book begins with the fundamental ideas of how one models experiments and measurements with macrosystems and eventually leads into a development of dual pairs of vector spaces to represent some

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of these concepts. Base-normed and order-unit normed dual pairs and their unit ball face lattices are introduced in the fourth chapter and it is here that our contribution begins. This brief note only touches upon a very small part of Ludwig's extensive work and our emphasis is on the functional analytic-geometric aspects of his work. Ensembles (states) and effects are embedded in these dual Banach spaces and decision effects are then introduced by Ludwig. Using several physically motivated axioms, it is shown that these decision effects form a complete orthomodular lattice in Chapter 6, Section 3. We start with a logic of propositions (an orthomodular poset called a quantum logic), carrying a large collection of states and we identify these propositions with Ludwig's decision effects. Most of our results are obtained from a careful investigation of the Hahn–Jordan decomposition theorem for states and it's consequences concerning the extreme points of the unit ball of the dual-order unit Banach space containing the quantum logic. At this point we analyze some of Ludwig's axioms and determine their consequences.

2. DEFINITIONS

A partially ordered ($\leq$) set Π with smallest element 0 and largest element e is called a *quantum logic*—or simply a *logic*-provided Π is *orthocomplemented, orthomodular*, and σ*-orthocomplete*. Each of these terms is defined: Orthocomplemented—For each $p \in \Pi$, there exists $p' \in \Pi$ such that $p \wedge p' = 0,\ p \vee p' = e,\ p'' = p$ and for all $p, q \in \Pi$, $p \leq q$ implies $q' \leq p'$. The element p' is called the *orthocomplement* of p. For each $p, q \in \Pi$ we call p and q *orthogonal* (denoted, $p \perp q$) in Π when $p \leq q'$, or equivalently $q \leq p'$. Next, we insist $p \vee q$ exists when $p \perp q$ and we denote this join as $p \oplus q$ in Π. Orthomodularity means: $p \leq q$ in Π implies there exists r in Π such that $p \perp r$ and $q = p \oplus r$. Lastly, σ-orthocomplete means: Each pair-wise orthogonal sequence (p_n) in Π has a supremum p in Π (denoted $p = \vee_n p_n$). Orthomodularity together with σ-orthocompleteness yields: Each monotone increasing sequence in Π has a supremum in Π. We emphasize here that Π need not be a lattice, so $\vee$ and $\wedge$ are not necessarily defined for all pairs in Π.

A function $\mu : \Pi \to [0, 1] \subset \mathbb{R}$ is called a *probability state*, or simply a *state*, provided $\mu(0) = 0, \mu(e) = 1,$ and for each orthogonal sequence (p_n) in $\Pi : \mu(\vee_n p_n) = \sum_{n=1}^{\infty} \mu(p_n)$. We will assume Π carries a large supply of states; specifically, let Ω represent the nonempty convex set of all states on Π and we insist Ω satisfies the following: For each $p \in \Pi$ ($p \neq 0$) there exists $\omega \in \Omega$ with $\omega(p) = 1$ and $p \leq q$ in Π iff for each $\omega \in \Omega,\ \omega(p) = 1$ implies $\omega(q) = 1$. When Ω satisfies this description we say Ω is a *strong* set of states.

We now generate dual linear spaces to contain Ω and Π. Let $\mathcal{F}(\Pi; \mathbb{R})$ represent the real, point-wise partially ordered, linear space of all real valued functions on Π and note that Ω is contained in its positive cone. Let $C = \{\alpha\omega : \alpha \geq 0$ in

$\mathbb{R}, \omega \in \Omega\}$ be the subcone with base Ω and define $V = C - C$ in $\mathcal{F}(\Pi; \mathbb{R})$. Next, let $U = \mathrm{con}(\Omega \cup -\Omega)$ (the convex hull of $\Omega \cup -\Omega$). The *base norm* on V (the Minkowski functional of U) makes V into a partially ordered Banach space (Cook, 1978). Specifically: for each $\mu \in V$, $\|\mu\| = \inf\{\alpha + \beta : \mu = \alpha\omega - \beta\nu, \ \alpha, \beta \geq 0 \text{ and } \omega, \nu \in \Omega\}$.

The base norm is called *minimal* if for each $\mu \in V$, there exist particular $\alpha, \beta \geq 0, \ \omega, \nu \in \Omega$ with $\mu = \alpha\omega - \beta\nu$ and $\|\mu\| = \alpha + \beta$. A minimal norm implies that the closed unit ball of V is exactly U. While this need not be the case in general, it is frequently observed to be true in physical examples. Henceforth, we will assume that Ω is strong over Π and the badse norm is minimal.

The ordered Banach dual of V is an order unit space (Alfsen, 1971, p. 78) and each $p \in \Pi$ can be treated as a positive linear functional $\hat{p}$ on V with the formula $\hat{p}(\mu) = \mu(p)$. When $p \neq 0, \ \|\hat{p}\| = 1$. The element $e \in \Pi$ is the order unit in V^* and the closed unit ball U^* is the order interval $[-e, e] = \{\varphi \in V^* : -e \leq \varphi \leq e\}$. We denote V^* with the dual-order unit norm by (L, e) or, simply L; further, the linear functional $\hat{p}$ will be denoted p and we will think of Π as a subset of $[0, e] \subset [-e, e]$. Since $[0, e]$ is $w(L, V)$-compact and convex, its extreme boundary is non-empty. It was shown in Cook (1978) that each $p \in \Pi$ is an extreme point of $[0, e]$ and Π is w^*-dense in the set of extreme points of $[0, e]$ when V satisfies the *ordered approximate Hahn–Jordan decomposition* property. This means: for each $\mu \in V$ and each $\varepsilon > 0$ there exist $\alpha, \beta \geq 0$ in $\mathbb{R}, \ \omega, \nu \in \Omega$ and $p \in \Pi$ such that $\mu = \alpha\omega - \beta\nu$, $\|\mu\| = \alpha + \beta$ and $\omega(p)$ and $\nu(p') < \varepsilon$. This condition implies $\mathrm{con}(\Pi)$ is w^*-dense in $[0, e]$, i.e., each $\varphi \in [0, e]$ is the w^*-limit of a net (φ_α) in $\mathrm{con}(\Pi)$. Also, the base norm is minimal. In order to show each p is extremal in $[0, e]$, it is sufficient to know that each φ_α can be chosen so that $0 \leq \varphi_\alpha \leq \varphi$ in the net described above. When this condition holds for the net (φ_α) we say V satisfies the *ordered* approximate Hahn–Jordan property. Using orthocomplementation on Π, if we can choose a net weakly approximating φ from below in $\mathrm{con}(\Pi)$, we can choose a net in $\mathrm{con}(\Pi)$ weakly approximating φ from above.

Our final preliminary idea is the following: For $A \subseteq \Omega$ ($B \subset [0, e]$) we define $A^1 = \{\varphi \in [0, e] : \omega(\varphi) = 1, \text{ for all } \omega \in A\}$ ($B_1 = \{\omega \in \Omega : \omega(\varphi) = 1, \text{ for all } \varphi \in B\}$). We note each A^1 is a $w(L, V)$-closed and *semiexposed* face of $[0, e]$ (B_1 is a $w(V, L)$- closed and semiexposed face of Ω). Recall that a semiexposed face is the intersection of the supporting hyperplanes which contain the face. The 1-maps defined here carry the complete Boolean algebra of all subsets of Ω (subsets of $[0, e]$) into the lattice of weak*-closed faces of $[0, e]$ (weakly closed faces of Ω). This is a familiar situation in lattice theory. For each $A \subset \Omega$, the correspondence $A \to {A^1}_1$, defines a *closure-operator* on the power set of Ω (similarly on $[0, e]$) and if $A = {A^1}_1$, A is said to be *Galois closed*. The pair of maps $((\cdot)^1, (\cdot)_1)$ defines a *Galois connection* of these power sets and it follows that the complete lattices of Galois closed subsets in Ω and $[0, e]$ form dually order-isomorphic complete lattices. These ideas are carefully and completely explained in Davey and Priestley (1990, pp. 232–233). We now want to connect the concepts: Lattices of Galois closed faces, the Hahn–Jordan decomposition property, and a strong set of states.

When Ω is strong over Π the correspondence of $p \to (p)_1 = \{\omega \in \Omega : \omega(p) = 1\}$ is a one-to-one and order preserving map of Π into the complete lattice of Galois closed faces of Ω. We denote the image of Π as $(\Pi)_1$. Note: $(0)_1 = \phi,\ p \leq q$ in Π iff $(p)_1 \subseteq (q)_1, (e)_1 = \Omega$. Further, the correspondence $p \to (p)_1^1$ is an order inverting map of Π into the complete lattice of Galois closed faces of $[0, e]$ ($(0)_1^1 = [0, e],\ p \leq q$ in Π iff $(q)_1^1 \subseteq (p)_1^1,\ (e)_1^1 = \{e\}$).

Theorem 2.1. *If Ω is strong over Π and V has the ordered approximate Hahn–Jordan decomposition property then:*

1. *Each $p \in \Pi$ is extremal in* $[0, e]$
2. *Π is w^*-dense in ext* $[0, e]$
3. *Each $p \in \Pi$ is the least element in $(p)_1^1$ and, thus, extremal in $(p)_1^1$. Further, $(p)_1^1 = [p, e]$.*
4. *Each $q \in \Pi \cap (p)_1^1$ is extremal in $(p)_1^1$ and $(p)_1^1 = w^*$-closure of con$(\Pi \cap (p)_1^1)$.*
5. *If $\varphi \in [0, e]$ and $(\varphi)_1^1 = [\varphi, e]$ then $\Pi \cap [\varphi, e]$ is w^*-dense in ext $[\varphi, e]$ and when V is* separable $\varphi \in \Pi$.

Proof. (1) and (2) are established in Cook (1978); (3) and (4) include and extend (1) and (2). To establish (3) let $\varphi \in (p)_1^1$. There exists a net (φ_α) in con(Π) with each $\varphi_\alpha \geq \varphi$ and the w^*-lim$(\varphi_\alpha) = \varphi$. For each $\omega \in (p)_1$ implies $\omega(\varphi) = 1$ implies $\omega(\varphi_\alpha) = 1$, for all α. Strong implies $\varphi_\alpha \geq p$ for each α, and therefore $\varphi \geq p$. So p is the least element in $(p)_1^1$ and $(p)_1^1 \subset [p, e]$. Thus, $(p)_1^1 = [p, e]$.

Since $q \in \Pi$ is extremal in $[0, e],\ q \in \Pi \cup (p)_1^1$ is extremal in $(p)_1^1$. If (φ_α) is a net in con(Π) as in the previous paragraph, each $\varphi_\alpha \in (p)_1^1$ and each $(p)_1^1$ is convex and w^*-compact. The Krein–Milman Theorem asserts ext$(p)_1^1$ is contained in the w^*-closure of $\Pi \cap (p)_1^1$ and (4) is established.

The first part of (5) is similar to (3) and (4). Recall that when V is separable the semiexposed faces of U and U^* (thus, Ω and $[0, e]$) are exposed (Ludwig, 1985, p. 121) and $[0, e]$ is a separable, compact, metric space in the w^*-topology. So, for each $\varphi \in [0, e],\ [\varphi, e]$ is also w^*-compact, separable, and metrizable. Therefore, let $(p_n) \subset \Pi$ be a *countable* maximal chain in $[\varphi, e]$ with the partial order inherited from L. By σ-orthocompleteness of Π, there exists $p \in \Pi$ with $p = \wedge_n p_n$ and p is also the w^*-limit of (p_n). So $p \in [\varphi, e]$ and $p \geq \varphi$. If $p \neq \varphi$, the ordered approximate Hahn–Jordan property implies there exist $\alpha > 0$ in $\mathbb{R}$ and $q \in \Pi$ such that $p - \varphi \geq \alpha q > 0$. Ω being strong implies $p \geq q$ and orthomodularity yields the existence of $r \in \Pi$ such that $p = q \oplus r$. For

$\omega \in (\varphi)_1$, $p \geq q$ implies $\omega(p) = \omega(\varphi) = 1$ and $\omega(q) = 0$; so, $\omega(r) = 1$. Hence, $r \in (\varphi)_1^1 = [\varphi, e]$ and $p \geq r$. This denies the maximality of (p_n), so $p = \varphi \in \Pi$.

Thus we have shown that when V is separable each $p \in \Pi$ is the least element in $(p)_1^1$ and each Galois closed interval $[\varphi, e]$ has its least element in Π. Ludwig calls these least elements *decision effects* (Ludwig, 1985, p. 160) and we have identified these with the elements of Π.

In the light of Theorem 2.1, let us return to the embedding of Π onto $(\Pi)_1$ in the Galois closed faces of Ω. For each $p \in \Pi$, $(p)_1 \cap (p')_1 = \phi$ (clear) and $(p)_1 \vee (p')_1 = \Omega$. $(((p)_1 \vee (p')_1)^1 = (p)_1^1 \cap (p')_1^1 = [p, e] \cap [p', e] = e$ in $[0, e]$ follows from strong and the Hahn–Jordan property.) Therefore, $(p')_1$ acts as a complement in $(\Pi)_1$, as expected. Further, we can define $(p)_1' = (p')_1$ and observe that $(p)_1'' = (p)_1$ and $(p)_1 \subseteq (q)_1$ implies $(q)_1' \subseteq (p)_1'$. Thus $(\Pi)_1$ is orthocomplemented. Orthogonality is introduced in $(\Pi)_1$ in the obvious fashion: $(p)_1 \perp (q)_1$ iff $(p)_1 \leq (q)_1'$ iff $p \leq q'$ iff $p \perp q$ in Π. When $p \perp q$ in Π $(p \oplus q)_1 = (p)_1 \vee (q)_1$. $(((p)_1 \vee (q)_1)^1 = (p)_1^1 \wedge (q)_1^1 = [p, e] \cap [q, e] = [p \oplus q, e]$; so, $(p)_1 \vee (q)_1 = (p \oplus q)_1$). Orthomodularity follows: $(p)_1 \subseteq (q)_1$ iff there exists $r \in \Pi$ with $q = p \oplus r$ which implies $(q)_1 = (p)_1 \vee (r)_1$. Likewise, σ-orthocompleteness follows in $(\Pi)_1$.

Caveat! We have *not* shown, even if Π is a lattice, that $(p \wedge q)_1 = (p)_1 \wedge (q)_1$, this last $\wedge$ being computed in the Galois closed faces. The example we present at the end of the paper illustrates this point.

Theorem 2.2. *If V is separable then: 1*

1. *For $\omega \in \Omega$, ω is an exposed point of U iff $\omega = (\omega)_1^1$.*
2. *For $\omega \in \Omega$, ω exposed in U implies $(\omega)^1$ is a maximal Galois closed face of $[0, e]$.*
3. *If $\varphi \in [0, e]$ and $[\varphi, e]$ is a maximal Galois closed face in $[0, e]$, then φ is an atom in Π.*

Proof: For each $\omega \in \Omega$, $(\omega)_1^1$ is semiexposed in U. Separability implies there exists $\varphi_0 \in [0, e]$ which exposes $(\omega)_1^1$ $((\varphi_0)_1 = (\omega)_1^1)$. Therefore, $\omega = (\omega)_1^1$ iff $\omega = (\varphi_0)_1$, so (1) holds. To establish (2) let $\omega \in \Omega$, $F = F_1^1$ a face with $(\omega)^1 \subset F$; we have $F_1 \subset (\omega)_1^1$. Next $\omega = (\omega)_1^1$ implies $F_1 = \omega$ which implies $F = F_1^1 = (\omega)^1$; so, $(\omega)^1$ is maximal. To obtain (3), Theorem 1 shows that $\varphi \in \Pi$. If there were $p \in \Pi$ with $0 < p < \varphi$ then $[p, e]$ being Galois closed would imply $[\varphi, e]$ is not maximal. □

Recall that L_1 the Banach lattice of all Lebesgue absolutely integrable functions on the unit interval is separable and its unit ball has no extreme points. Thus, the existence of exposed points in the unit ball is not always obtained. We consider this problem in a forthcoming paper.

Let us now turn to Ludwig's axiom AV1.1 (Ludwig, 1985, p. 155) and phrase it, equivalently, in the form of this note.

Axiom AV1.1 For each pair g_1, g_2 in $[0, e]$ there exists $g \in [0, e]$ such that

(1) $g \leq g_1, g_2$, and
(2) $(g_1)_1 \cap (g_2)_1 = (g)_1$.

The physical justification of this axiom is given in Chapter 6, Section 1. We simply examine its mathematical consequences.

Theorem 2.3. *If we assume AV 1.1, each Galois closed face $F \subset [0, e]$ has a least element. When V is separable this least element is in Π.*

Proof: Let $F = F_1^1$, (φ_α) a maximal chain in F (Zorn's lemma) and let $\varphi = \bigwedge_\alpha(\varphi_\alpha)$. Then, φ is the w^*-limit of (φ_α); so $\varphi \in F$. We assert φ is the least element in F: If $\psi \in F$, AV 1.1 asserts the existence of $g \in [0, e]$ with $g \leq \varphi, \psi$ and $(g)_1 = (\varphi)_1 \cap (\psi)_1$. Thus, $\omega \in F_1$ implies $\varphi(\omega) = \psi(\omega) = 1$ implies $g(\omega) = 1$ implies $g \in F_1^1 = F$ implies $g = \varphi$ implies $\varphi \leq \psi$ and $F = [\varphi, e]$. Theorem 1. (5) now asserts when V is separable $\varphi \in \Pi$. □

Corollary 2.4. *When V is separable and AV 1.1 holds Π is a σ-complete orthomodular lattice which is isomorphic to each of the lattices of Galois closed faces described above.*

Proof: For p, q in Π, AV 1.1 yields $g \in [0, e]$ with $g \leq p, q$ and $(g)_1 = (p)_1 \cap (q)_1$. This implies $(g)_1^1 = [g, e]$ is Galois closed in $[0, e]$, so $g \in \Pi$. If $\varphi \in \Pi$ and $\varphi \leq p, q$ then $(\varphi)_1 \subseteq (g)_1$ implies $\varphi \leq g$ from Ω being strong. Thus g must be $p \wedge q$. The remainder follows directly. □

An observation connecting the work of Alfsen and Schultz (1976) to this result is as follows: No linear P—projections on the base-normed or order-unit normed spaces have been introduced here and yet the lattices of Galois closed faces and decision effects have been shown to be isomorphic. An expanded development of this work is presently underway in which the necessity of separability in V, the existence of exposed points in Ω and in $[0, e]$, the roles of dual and predual Banach spaces, and spaces of finitely additive states will be integrated into this material.

Some additional brief historical remarks connecting this note with past works are appropriate here. Rüttimann defined the notion of the approximate Jordan–Hahn property in Rüttimann (1989) and obtained our Theorem 2.1 parts (1) and

(2) for finitely additive states. He also obtained results concerning the equivalence of the total variation norm and the base norm similar to the results in Cook (1978). Of course, there is a good deal more in the Rüttimann paper than we have touched upon here. The addition of the *ordered* part in our definition of the approximate Hahn–Jordan property is sufficient (and not very strong) to obtain that each element of Π is extremal in $[0, e]$. Most examples encountered in physical situations satisfy a form of the spectral theorem or a form of the monotone convergence theorem and our ordered approximate Hahn–Jordan property is easily satisfied. Keller (1989a) proves that Ω strong, Π an orthoposet, and $\Pi = \text{ext}\,[0, e]$ yield Π is an $\perp$- complete orthomodular lattice and V satisfies the ε—Hahn–Jordan property (Cook, 1978). He also shows in this paper that Π being w^*—compact together with the ε—Hahn–Jordan property is sufficient to guarantee $\Pi = \text{ext}\,[0, e]$. In Keller (1989b) generalizes some earlier work of Rüttimann on Jauch–Piron states and obtains the result that the ext $[0, e]$ forms an orthomodular lattice when the elements of Ω are Jauch-Piron. Under various hypotheses on Ω and Π, Ω being a Jauch-Piron state space is essentially the assumption of Ludwig's axiom AV 1.1. For the definitions and details the reader must consult these papers.

Finally, we conclude with a simple example, called the "Bow Tie Manual" (Foulis and Randall, 1972). In this example AV 1.1 is *not* satisfied, the logic is a non-Boolean orthomodular lattice, the probability states are strong, the positive exposed points of U are equal in number to the atoms in Π, and the map $p \to (p)_1$ is *not* a lattice isomorphism. This example and several others like it can be found in Dvurečenskij and Pulmannová (2000) in Section 4.3.3, pages 269–274.

Consider an experiment of two non-communicating observers E and F who watch a fire fly with a randomly flashing light in a box. E observes: no light

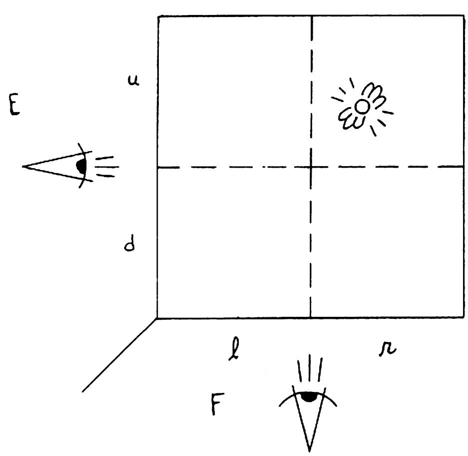

Fig. 1. The experiment.

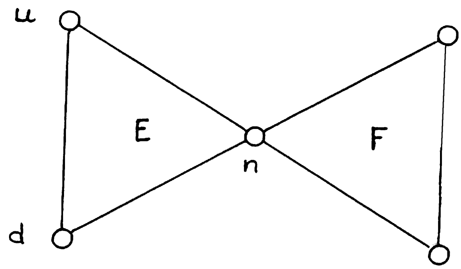

Fig. 2. Outcome-orthogonality.

(n), light on in up-position (u), or light on in down-position (d). F observes analogously and we agree that no light (n) is the same for both observers (Fig. 1). The dotted lines represent porous boundaries which the fly passes through without noticing.

The outcome-orthogonality set of this experiment is $X = \{u, d, n, \ell, r\}$ and is represented in Fig. 2. The nodes represent outcomes and the lines connecting the nodes physical rejection or orthogonality.

The logic Π of this experiment is given by Fig. 3. The nodes represent propositions and the lines logical implication. Figure 4 represents the convex set Ω. For example: $\omega_{\ell,d}(p(d)) = 1, \omega_{\ell,d}(p(r)) = 0$, etc. Note that $\{\ell, d\}$ is not a testable event because E and F are non-communicating.

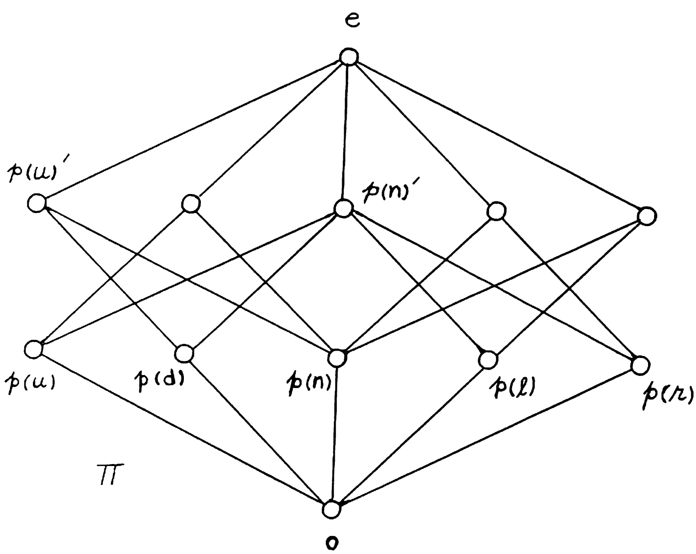

Fig. 3. The logic.

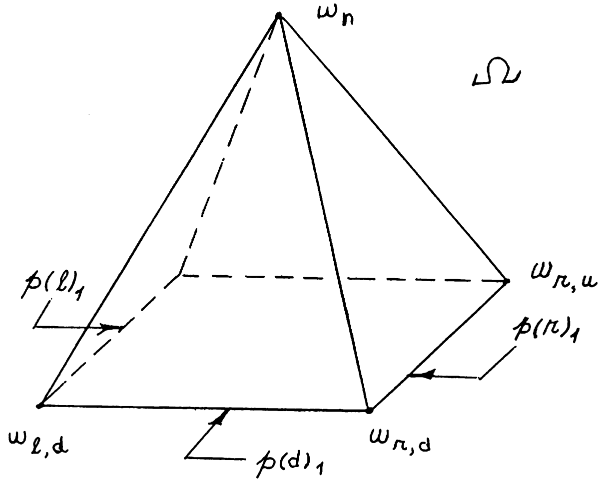

Fig. 4. Probablity states.

The base-normed space V is four-dimensional, the base Ω is a three-dimensional polytope which is *not* a simplex, so Π is non-Boolean, $p(d)$ and $p(\ell)$ are atoms in Π, $p(\ell)_1$ and $p(d)_1$ (see Fig. 4) are smallest non-zero elements in $(\Pi)_1$ but are not atoms in the lattice of Galois closed faces of Ω. Observe that $(\omega_{(\ell,d)})^1$ is a maximal face in the Galois closed faces of $[0, e]$, but has *no* least element. Both $p(d)$ and $p(\ell)$ are minimal elements in $(\omega_{\ell,d})^1$ which has, in fact, infinitely many minimal elements. Finally, $(\omega_{\ell,d})^1_1 = \omega_{\ell,d}$ (Theorem 2.2 (1)) but is not in $(\Pi)_1$.

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